

**What Is Claimed Is:**

- 1           1.       A method for using a computer system to solve a system of  
2 nonlinear equations specified by a vector function,  $\mathbf{f}$ , wherein  $\mathbf{f}(\mathbf{x}) = \mathbf{0}$  represents a  
3 set of nonlinear equations,  $f_1(\mathbf{x}) = 0, f_2(\mathbf{x}) = 0, f_3(\mathbf{x}) = 0, \dots, f_n(\mathbf{x}) = 0$ , wherein  $\mathbf{x}$   
4 is a vector  $(x_1, x_2, x_3, \dots, x_n)$ , the method comprising:  
5           receiving a representation of a subbox  $\mathbf{X} = (X_1, X_2, \dots, X_n)$ , wherein for  
6 each dimension,  $i$ , the representation of  $X_i$  includes a first floating-point number,  
7  $a_i$ , representing the left endpoint of  $X_i$ , and a second floating-point number,  $b_i$ ,  
8 representing the right endpoint of  $X_i$ ;  
9           storing the representation in a computer memory;  
10          applying term consistency to the set of nonlinear equations,  $f_1(\mathbf{x}) = 0$ ,  
11  $f_2(\mathbf{x}) = 0, f_3(\mathbf{x}) = 0, \dots, f_n(\mathbf{x}) = 0$ , over  $\mathbf{X}$ , and excluding portions of  $\mathbf{X}$  that violate  
12 any of these nonlinear equations;  
13          applying box consistency to the set of nonlinear equations over  $\mathbf{X}$ , and  
14 excluding portions of  $\mathbf{X}$  that violate any of the nonlinear equations; and  
15          performing an interval Newton step on  $\mathbf{X}$  to produce a resulting subbox  $\mathbf{Y}$ ,  
16 wherein the point of expansion of the interval Newton step is a point  $\mathbf{x}$  within  $\mathbf{X}$ ,  
17 and wherein performing the interval Newton step involves evaluating  $\mathbf{f}(\mathbf{x})$  using  
18 interval arithmetic to produce an interval result  $\mathbf{f}^I(\mathbf{x})$ .
- 1           2.       The method of claim 1, wherein performing the interval Newton  
2 step involves:  
3           computing  $\mathbf{J}(\mathbf{x}, \mathbf{X})$ , wherein  $\mathbf{J}(\mathbf{x}, \mathbf{X})$  is the Jacobian of the function  $\mathbf{f}$   
4 evaluated as a function of  $\mathbf{x}$  over the subbox  $\mathbf{X}$ ; and  
5           determining if  $\mathbf{J}(\mathbf{x}, \mathbf{X})$  is regular as a byproduct of solving for the subbox  $\mathbf{Y}$   
6 that contains values of  $\mathbf{y}$  that satisfy  $\mathbf{M}(\mathbf{x}, \mathbf{X})(\mathbf{y} - \mathbf{x}) = \mathbf{r}(\mathbf{x})$ , where

1  $\mathbf{M}(\mathbf{x}, \mathbf{X}) = \mathbf{B}\mathbf{J}(\mathbf{x}, \mathbf{X})$ ,  $\mathbf{r}(\mathbf{x}) = -\mathbf{B}\mathbf{f}(\mathbf{x})$ , and  $\mathbf{B}$  is an approximate inverse of the center of  
2  $\mathbf{J}(\mathbf{x}, \mathbf{X})$ .

1 3. The method of claim 2, further comprising:  
2 applying term consistency to the preconditioned set of nonlinear equations  
3  $\mathbf{B}\mathbf{f}(\mathbf{x}) = \mathbf{0}$  over the subbox  $\mathbf{X}$ ; and  
4 excluding portions of  $\mathbf{X}$  that violate the preconditioned set of nonlinear  
5 equations.

1 4. The method of claim 2, further comprising:  
2 applying box consistency to the preconditioned set of nonlinear equations  
3  $\mathbf{B}\mathbf{f}(\mathbf{x}) = \mathbf{0}$  over the subbox  $\mathbf{X}$ ; and  
4 excluding portions of  $\mathbf{X}$  that violate the preconditioned set of nonlinear  
5 equations.

1 5. The method of claim 1, wherein applying term consistency to the  
2 set of nonlinear equations involves:  
3 for each nonlinear equation  $f_i(\mathbf{x}) = 0$  in the system of equations  $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ ,  
4 symbolically manipulating  $f_i(\mathbf{x}) = 0$  to solve for an invertible term,  $g(x'_j)$ , thereby  
5 producing a modified equation  $g(x'_j) = h(\mathbf{x})$ , wherein  $g(x'_j)$  can be analytically  
6 inverted to produce an inverse function  $g^{-1}(\mathbf{y})$ ;  
7 substituting the subbox  $\mathbf{X}$  into the modified equation to produce the  
8 equation  $g(X'_j) = h(\mathbf{X})$ ;  
9 solving for  $X'_j = g^{-1}(h(\mathbf{X}))$ ; and  
10 intersecting  $X'_j$  with the vector element  $X_j$  to produce a new subbox  $\mathbf{X}^+$ ;

11 wherein the new subbox  $\mathbf{X}^+$  contains all solutions of the system of  
12 equations  $\mathbf{f}(\mathbf{x}) = \mathbf{0}$  within the subbox  $\mathbf{X}$ , and wherein the width of the new subbox  
13  $\mathbf{X}^+$  is less than or equal to the width of the subbox  $\mathbf{X}$ .

1 6. The method of claim 1, further comprising:  
2 evaluating a first termination condition, wherein the first termination  
3 condition is TRUE if,  
4 zero is contained within  $\mathbf{f}^1(\mathbf{x})$ ,  
5  $\mathbf{J}(\mathbf{x}, \mathbf{X})$  is regular, wherein  $\mathbf{J}(\mathbf{x}, \mathbf{X})$  is the Jacobian of the  
6 function  $\mathbf{f}$  evaluated as a function of  $\mathbf{x}$  over the subbox  $\mathbf{X}$ , and  
7 the solution  $\mathbf{Y}$  of  $\mathbf{M}(\mathbf{x}, \mathbf{X}) (\mathbf{y} - \mathbf{x}) = \mathbf{r}$  contains  $\mathbf{X}$ ; and  
8 if the first termination condition is TRUE, terminating and recording  $\mathbf{X}$  as  
9 a final bound.

1 7. The method of claim 6, wherein the method further comprises:  
2 evaluating a second termination condition;  
3 wherein the second termination condition is TRUE if a function of the  
4 width of the subbox  $\mathbf{X}$  is less than a pre-specified value,  $\varepsilon_X$ , and the width of the  
5 function  $\mathbf{f}$  over the subbox  $\mathbf{X}$  is less than a pre-specified value,  $\varepsilon_F$ ; and  
6 if the second termination condition is TRUE, terminating and recording  $\mathbf{X}$   
7 as a final bound.

1 8. A computer-readable storage medium storing instructions that  
2 when executed by a computer cause the computer to perform a method for using a  
3 computer system to solve a system of nonlinear equations specified by a vector  
4 function,  $\mathbf{f}$ , wherein  $\mathbf{f}(\mathbf{x}) = \mathbf{0}$  represents a set of nonlinear equations,  $f_i(\mathbf{x}) = 0$ ,

5  $f_2(\mathbf{x}) = 0, f_3(\mathbf{x}) = 0, \dots, f_n(\mathbf{x}) = 0$ , wherein  $\mathbf{x}$  is a vector  $(x_1, x_2, x_3, \dots, x_n)$ , the  
 6 method comprising:  
 7 receiving a representation of a subbox  $\mathbf{X} = (X_1, X_2, \dots, X_n)$ , wherein for  
 8 each dimension,  $i$ , the representation of  $X_i$  includes a first floating-point number,  
 9  $a_i$ , representing the left endpoint of  $X_i$ , and a second floating-point number,  $b_i$ ,  
 10 representing the right endpoint of  $X_i$ ;  
 11 storing the representation in a computer memory;  
 12 applying term consistency to the set of nonlinear equations,  $f_1(\mathbf{x}) = 0$ ,  
 13  $f_2(\mathbf{x}) = 0, f_3(\mathbf{x}) = 0, \dots, f_n(\mathbf{x}) = 0$ , over  $\mathbf{X}$ , and excluding portions of  $\mathbf{X}$  that violate  
 14 any of these nonlinear equations;  
 15 applying box consistency to the set of nonlinear equations over  $\mathbf{X}$ , and  
 16 excluding portions of  $\mathbf{X}$  that violate any of the nonlinear equations; and  
 17 performing an interval Newton step on  $\mathbf{X}$  to produce a resulting subbox  $\mathbf{Y}$ ,  
 18 wherein the point of expansion of the interval Newton step is a point  $\mathbf{x}$  within  $\mathbf{X}$ ,  
 19 and wherein performing the interval Newton step involves evaluating  $\mathbf{f}(\mathbf{x})$  using  
 20 interval arithmetic to produce an interval result  $\mathbf{f}^1(\mathbf{x})$ .

1 9. The computer-readable storage medium of claim 8, wherein  
 2 performing the interval Newton step involves:  
 3 computing  $\mathbf{J}(\mathbf{x}, \mathbf{X})$ , wherein  $\mathbf{J}(\mathbf{x}, \mathbf{X})$  is the Jacobian of the function  $\mathbf{f}$   
 4 evaluated as a function of  $\mathbf{x}$  over the subbox  $\mathbf{X}$ ; and  
 5 determining if  $\mathbf{J}(\mathbf{x}, \mathbf{X})$  is regular as a byproduct of solving for the subbox  $\mathbf{Y}$   
 6 that contains values of  $\mathbf{y}$  that satisfy  $\mathbf{M}(\mathbf{x}, \mathbf{X})(\mathbf{y} - \mathbf{x}) = \mathbf{r}(\mathbf{x})$ , where  
 7  $\mathbf{M}(\mathbf{x}, \mathbf{X}) = \mathbf{B}\mathbf{J}(\mathbf{x}, \mathbf{X})$ ,  $\mathbf{r}(\mathbf{x}) = -\mathbf{B}\mathbf{f}(\mathbf{x})$ , and  $\mathbf{B}$  is an approximate inverse of the center of  
 8  $\mathbf{J}(\mathbf{x}, \mathbf{X})$ .

1           10.     The computer-readable storage medium of claim 9, wherein the  
 2 method further comprises:  
 3           applying term consistency to the preconditioned set of nonlinear equations  
 4  $\mathbf{Bf}(\mathbf{x}) = \mathbf{0}$  over the subbox  $\mathbf{X}$ ; and  
 5           excluding portions of  $\mathbf{X}$  that violate the preconditioned set of nonlinear  
 6 equations.

1           11.     The computer-readable storage medium of claim 9, wherein the  
 2 method further comprises:  
 3           applying box consistency to the preconditioned set of nonlinear equations  
 4  $\mathbf{Bf}(\mathbf{x}) = \mathbf{0}$  over the subbox  $\mathbf{X}$ ; and  
 5           excluding portions of  $\mathbf{X}$  that violate the preconditioned set of nonlinear  
 6 equations.

1           12.     The computer-readable storage medium of claim 8, wherein  
 2 applying term consistency to the set of nonlinear equations involves:  
 3           for each nonlinear equation  $f_i(\mathbf{x}) = 0$  in the system of equations  $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ ,  
 4 symbolically manipulating  $f_i(\mathbf{x}) = 0$  to solve for an invertible term,  $g(x'_j)$ , thereby  
 5 producing a modified equation  $g(x'_j) = h(\mathbf{x})$ , wherein  $g(x'_j)$  can be analytically  
 6 inverted to produce an inverse function  $g^{-1}(\mathbf{y})$ ;  
 7           substituting the subbox  $\mathbf{X}$  into the modified equation to produce the  
 8 equation  $g(X'_j) = h(\mathbf{X})$ ;  
 9           solving for  $X'_j = g^{-1}(h(\mathbf{X}))$ ; and  
 10          intersecting  $X'_j$  with the vector element  $X_j$  to produce a new subbox  $\mathbf{X}^+$ ;  
 11          wherein the new subbox  $\mathbf{X}^+$  contains all solutions of the system of  
 12 equations  $\mathbf{f}(\mathbf{x}) = \mathbf{0}$  within the subbox  $\mathbf{X}$ , and wherein the width of the new subbox  
 13  $\mathbf{X}^+$  is less than or equal to the width of the subbox  $\mathbf{X}$ .

1           13.     The computer-readable storage medium of claim 8, wherein the  
2     method further comprises:  
3           evaluating a first termination condition, wherein the first termination  
4     condition is TRUE if,  
5                     zero is contained within  $\mathbf{f}^l(\mathbf{x})$ ,  
6                      $\mathbf{J}(\mathbf{x}, \mathbf{X})$  is regular, wherein  $\mathbf{J}(\mathbf{x}, \mathbf{X})$  is the Jacobian of the  
7           function  $\mathbf{f}$  evaluated as a function of  $\mathbf{x}$  over the subbox  $\mathbf{X}$ , and  
8                     the solution  $\mathbf{Y}$  of  $\mathbf{M}(\mathbf{x}, \mathbf{X}) (\mathbf{y} - \mathbf{x}) = \mathbf{r}$  contains  $\mathbf{X}$ ; and  
9           if the first termination condition is TRUE, terminating and recording  $\mathbf{X}$  as  
10    a final bound.

1           14.     The computer-readable storage medium of claim 13, wherein the  
2     method further comprises:  
3           evaluating a second termination condition;  
4           wherein the second termination condition is TRUE if a function of the  
5     width of the subbox  $\mathbf{X}$  is less than a pre-specified value,  $\varepsilon_X$ , and the width of the  
6     function  $\mathbf{f}$  over the subbox  $\mathbf{X}$  is less than a pre-specified value,  $\varepsilon_F$ ; and  
7           if the second termination condition is TRUE, terminating and recording  $\mathbf{X}$   
8     as a final bound.

1           15.     An apparatus that solves a system of nonlinear equations specified  
2     by a vector function,  $\mathbf{f}$ , wherein  $\mathbf{f}(\mathbf{x}) = \mathbf{0}$  represents a set of nonlinear equations,  
3      $f_1(\mathbf{x}) = 0, f_2(\mathbf{x}) = 0, f_3(\mathbf{x}) = 0, \dots, f_n(\mathbf{x}) = 0$ , wherein  $\mathbf{x}$  is a vector  $(x_1, x_2, x_3, \dots, x_n)$ ,  
4     the apparatus comprising:  
5           a receiving mechanism that is configured to receive a representation of a  
6     subbox  $\mathbf{X} = (X_1, X_2, \dots, X_n)$ , wherein for each dimension,  $i$ , the representation of

7  $X_i$  includes a first floating-point number,  $a_i$ , representing the left endpoint of  $X_i$ ,  
 8 and a second floating-point number,  $b_i$ , representing the right endpoint of  $X_i$ ;  
 9 a computer memory for storing the representation;  
 10 a term consistency mechanism that is configured to apply term consistency  
 11 to the set of nonlinear equations,  $f_1(\mathbf{x}) = 0, f_2(\mathbf{x}) = 0, f_3(\mathbf{x}) = 0, \dots, f_n(\mathbf{x}) = 0$ , over  
 12  $\mathbf{X}$ , and to exclude portions of  $\mathbf{X}$  that violate any of these nonlinear equations;  
 13 a box consistency mechanism that is configured to apply box consistency  
 14 to the set of nonlinear equations over  $\mathbf{X}$ , and to exclude portions of  $\mathbf{X}$  that violate  
 15 any of the nonlinear equations; and  
 16 an interval Newton mechanism that is configured to perform an interval  
 17 Newton step on  $\mathbf{X}$  to produce a resulting subbox  $\mathbf{Y}$ , wherein the point of  
 18 expansion of the interval Newton step is a point  $\mathbf{x}$  within  $\mathbf{X}$ , and wherein  
 19 performing the interval Newton step involves evaluating  $\mathbf{f}(\mathbf{x})$  using interval  
 20 arithmetic to produce an interval result  $\mathbf{f}^I(\mathbf{x})$ .

1 16. The apparatus of claim 15, wherein the interval Newton  
 2 mechanism is configured to:  
 3 compute  $\mathbf{J}(\mathbf{x}, \mathbf{X})$ , wherein  $\mathbf{J}(\mathbf{x}, \mathbf{X})$  is the Jacobian of the function  $\mathbf{f}$  evaluated  
 4 as a function of  $\mathbf{x}$  over the subbox  $\mathbf{X}$ ; and  
 5 determine if  $\mathbf{J}(\mathbf{x}, \mathbf{X})$  is regular as a byproduct of solving for the subbox  $\mathbf{Y}$   
 6 that contain the values of  $\mathbf{y}$  that satisfy  $\mathbf{M}(\mathbf{x}, \mathbf{X})(\mathbf{y} - \mathbf{x}) = \mathbf{r}(\mathbf{x})$ , where  
 7  $\mathbf{M}(\mathbf{x}, \mathbf{X}) = \mathbf{B}\mathbf{J}(\mathbf{x}, \mathbf{X})$ ,  $\mathbf{r}(\mathbf{x}) = -\mathbf{B}\mathbf{f}(\mathbf{x})$ , and  $\mathbf{B}$  is an approximate inverse of the center of  
 8  $\mathbf{J}(\mathbf{x}, \mathbf{X})$ .

1 17. The apparatus of claim 16, wherein the term consistency  
 2 mechanism is configured to:

1           apply term consistency to the preconditioned set of nonlinear equations  
2     $\mathbf{Bf}(\mathbf{x}) = \mathbf{0}$  over the subbox  $\mathbf{X}$ ; and to  
3           exclude portions of  $\mathbf{X}$  that violate the preconditioned set of nonlinear  
4    equations.

1           18.    The apparatus of claim 16, wherein the box consistency  
2    mechanism is configured to:  
3           apply box consistency to the preconditioned set of nonlinear equations  
4     $\mathbf{Bf}(\mathbf{x}) = \mathbf{0}$  over the subbox  $\mathbf{X}$ ; and to  
5           exclude portions of  $\mathbf{X}$  that violate the preconditioned set of nonlinear  
6    equations.

1           19.    The apparatus of claim 15, wherein for each nonlinear equation  
2     $f_i(\mathbf{x}) = 0$  in the system of equations  $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ , the term consistency mechanism is  
3    configured to:  
4           symbolically manipulate  $f_i(\mathbf{x})=0$  to solve for an invertible term,  $g(x'_j)$ ,  
5    thereby producing a modified equation  $g(x'_j) = h(\mathbf{x})$ , wherein  $g(x'_j)$  can be  
6    analytically inverted to produce an inverse function  $g^{-1}(\mathbf{y})$ ;  
7           substitute the subbox  $\mathbf{X}$  into the modified equation to produce the equation  
8     $g(X'_j) = h(\mathbf{X})$ ;  
9           solve for  $X'_j = g^{-1}(h(\mathbf{X}))$ ; and to  
10          intersect  $X'_j$  with the vector element  $X_j$  to produce a new subbox  $\mathbf{X}^+$ ;  
11          wherein the new subbox  $\mathbf{X}^+$  contains all solutions of the system of  
12    equations  $\mathbf{f}(\mathbf{x}) = \mathbf{0}$  within the subbox  $\mathbf{X}$ , and wherein the width of the new subbox  
13     $\mathbf{X}^+$  is less than or equal to the width of the subbox  $\mathbf{X}$ .



1           20.    The apparatus of claim 15, further comprising a termination  
2 mechanism that is configured to:  
3           evaluate a first termination condition, wherein the first termination  
4 condition is TRUE if,  
5                       zero is contained within  $\mathbf{f}^l(\mathbf{x})$ ,  
6                        $\mathbf{J}(\mathbf{x}, \mathbf{X})$  is regular, wherein  $\mathbf{J}(\mathbf{x}, \mathbf{X})$  is the Jacobian of the  
7 function  $\mathbf{f}$  evaluated as a function of  $\mathbf{x}$  over the subbox  $\mathbf{X}$ , and  
8                       the solution  $\mathbf{Y}$  of  $\mathbf{M}(\mathbf{x}, \mathbf{X}) (\mathbf{y} - \mathbf{x}) = \mathbf{r}$  contains  $\mathbf{X}$ ; and to  
9 terminate and record  $\mathbf{X}$  as a final bound if the first termination condition is  
10 TRUE.

1           21.    The apparatus of claim 20, wherein the termination mechanism is  
2 additionally configured to:  
3           evaluate a second termination condition;  
4           wherein the second termination condition is TRUE if a function of the  
5 width of the subbox  $\mathbf{X}$  is less than a pre-specified value,  $\varepsilon_X$ , and the width of the  
6 function  $\mathbf{f}$  over the subbox  $\mathbf{X}$  is less than a pre-specified value,  $\varepsilon_F$ ; and to  
7 terminate and record  $\mathbf{X}$  as a final bound if the second termination  
8 condition is TRUE.